Local Uncertainty Principles for Signals on Graphs

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Abstract

Uncertainty principles such as Heisenberg's provide limits on the time-frequency concentration of a signal, and constitute an important theoretical tool for designing and evaluating linear signal transforms. Generalizations of such principles to the graph setting can inform dictionary design for graph signals, lead to algorithms for reconstructing missing information from graph signals via sparse representations, and yield new graph analysis tools. While previous work has focused on generalizing notions of spreads of a graph signal in the vertex and graph spectral domains, our approach is to generalize the methods of Lieb in order to develop uncertainty principles that provide limits on the concentration of the analysis coefficients of any graph signal under a dictionary transform whose atoms are jointly localized in the vertex and graph spectral domains. One challenge we highlight is that due to the inhomogeneity of the underlying graph data domain, the local structure in a single small region of the graph can drastically affect the uncertainty bounds for signals concentrated in different regions of the graph, limiting the information provided by global uncertainty principles. Accordingly, we suggest a new way to incorporate a notion of locality, and develop local uncertainty principles that bound the concentration of the analysis coefficients of each atom of a localized graph spectral filter frame in terms of quantities that depend on the local structure of the graph around the center vertex of the given atom. Finally, we demonstrate how our proposed local uncertainty measures can improve the random sampling of graph signals.

1 Motivation

Goal

Build an analysis tool able to assess of the structure of a graph.

- **Since graphs are irregular structure, we need to probe the local structure of a graph**

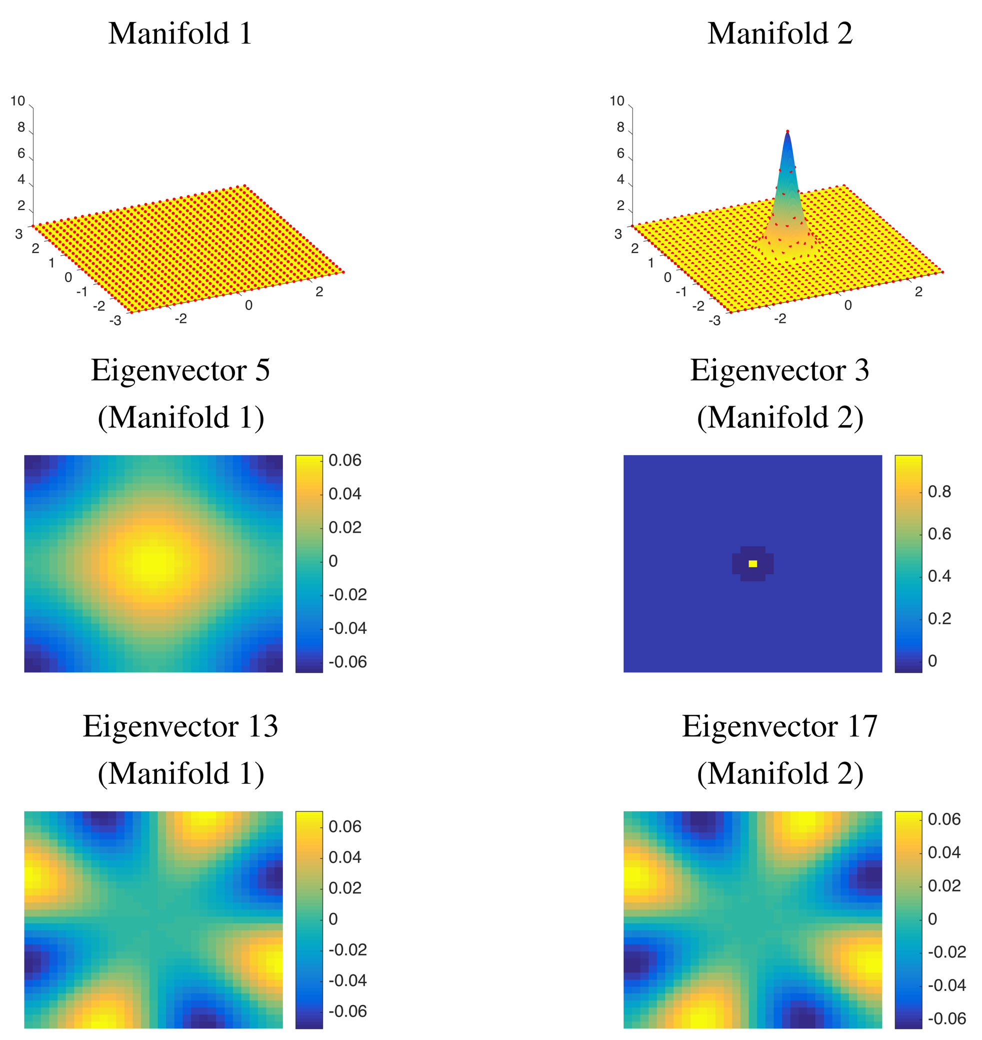


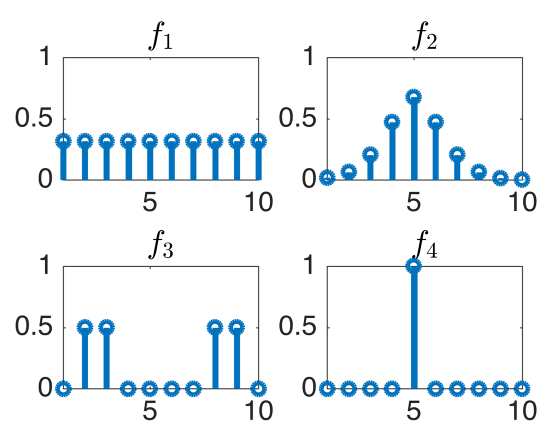
Figure 1: Concentration of graph Laplacian eigenvectors. We discretize two different manifolds by sampling uniformly across the x-y plane.

2 Characterization of the concentration

Concentration measure

Inspired by Lieb [4], we use ℓ^p-norms

$$s_p(f) = \begin{cases} \frac{\|f\|_2}{\|f\|_p}, & \text{if } 1 \le p \le 2 \\ \frac{\|f\|_p}{\|f\|_2}, & \text{if } 2 < p \le \infty \end{cases}.$$



p	$s_p(f_1)$	$s_p(f_2)$	$s_p(f_3)$	$s_p(f_4)$
1.00	0.32	0.46	0.50	1.00
1.33	0.56	0.69	0.71	1.00
2.00	1.00	1.00	1.00	1.00
4.00	0.56	0.75	0.71	1.00
∞	0.32	0.68	0.50	1.00

Figure 2: The concentration $s_p(\cdot)$ of four different example signals (all with 2-norm equal to 1), for various values of p.

3 Definition & notation

- u_ℓ, λ_ℓ are the eigenvectors and eigenvalues of the Laplacian $\mathcal{L}$ [2]
- Localize a kernel $\hat{g}$ to center vertex $i \in \{1, 2, \ldots, N\}$ by applying:

$$T_i g(n) = \sqrt{N} \sum_{\ell=0}^{N-1} \hat{g}(\lambda_\ell) \overline{u_\ell(i)} u_\ell(n)$$

- A collection of filters $\mathcal{D} = \{g_k\}_{k=1,\ldots,K}$ forms a graph filter frame if there exists A and B such that for all $f \in \mathbb{C}^N$ [3]:

$$A\|f\|_2^2 \le \sum_{i,k} |\langle f, T_i g_k \rangle|^2 \le B\|f\|_2^2.$$

- Special case: uniform translates of a a kernel (Gabor filterbank).
- Analysis operator:

$$\mathcal{A}_{\mathcal{D}} f(i,k) = \langle f, T_i g_k \rangle.$$

- Ambiguity function (Characterization of the frame):

$$\mathbb{A}_{\mathrm{g}}(i_0, k_0, i, k) = \mathcal{A}_{\mathrm{g}} T_{i_0} g_{k_0}(i,k) = \langle T_{i_0} g_{k_0}, T_i g_k \rangle$$

- Norm of the ambibuity function

$$\|\mathcal{A}_{\mathrm{g}} T_{i_0} g_{k_0}\|_p = \left(\sum_{k=1}^{M} \sum_{i=1}^{N} |< T_i g_k, T_{i_0} g_{k_0} >|^p \right)^{\frac{1}{p}}$$

4 Theorems

Theorem 1 (Global uncertainty). *Let* $\{T_i g\}_{\{i\in[1,N],k\in[0,M-1]\}}$ *be a graph filter frame. For any signal* $f \in \mathbb{C}^N$ *and for any* $p \in [1,\infty]$, *we have*

$$s_p(\mathcal{A}_{\mathrm{g}} f) \le \frac{B^{\min\{\frac{1}{2},\frac{1}{p}\}}}{A^{\max\{\frac{1}{2},\frac{1}{p}\}}} \left(\max_{i,k} \|T_i g_k\|_2 \right)^{\left|1-\frac{2}{p}\right|}. \qquad (1)$$

Local uncertainty

Theorem 2. *Let* $\{T_i g\}_{\{i\in[1,N],k\in[0,M-1]\}}$ *be a graph filter frame. For any* $i_0 \in [1,N]$, $k_0 \in [0, M-1]$ *such that* $\|T_{i_0} g_{k_0}\|_2 > 0$, *then for* $p \in [1,\infty]$, *we have*

$$s_p\left(\mathcal{A}_{\mathrm{g}} T_{i_0} g_{k_0}\right) \le \frac{B^{\min\{\frac{1}{p},1-\frac{1}{p}\}} \|T_{\tilde{i}_{i_0,k_0}} g_{\tilde{k}_{i_0,k_0}}\|_2^{\left|1-\frac{2}{p}\right|}}{A^{\frac{1}{2}}} \qquad (2)$$

where $\tilde{k}_{i_0,k_0} = \arg\max_k \|T_{i_0}(g_{k_0} \cdot g_k)\|_\infty$, *and* $\tilde{i}_{i_0,k_0} = \arg\max_i \left| T_{i_0}(g_{k_0} \cdot g_{\tilde{k}_{i_0,k_0}})(i) \right|$.

- i_0 **selects the node and** k_0 **the frequency band**
- $\tilde{i}_{i_0,k_0}$ is close to i_0 if the kernel g_k is smooth (true for Gabor)
- $\tilde{k}_{i_0,k_0}$ is close to k_0 if the kernel is localized (true for Gabor)

5 Illustration

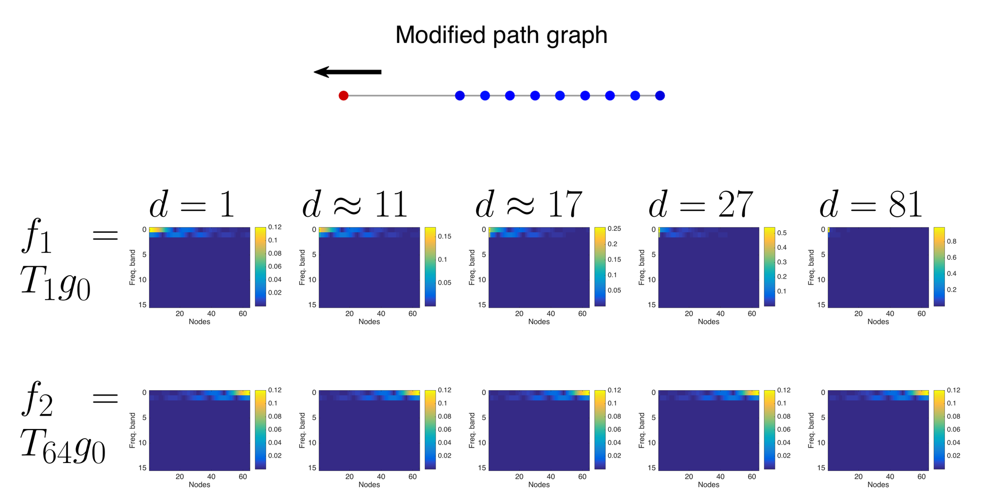


Figure 3: Graph Gabor transforms of $f_1 = T_1 g_0$ and $f_2 = T_{64} g_0$ for 5 different distances between vertices 1 and 2 of the modified path graph (64 vertices). The distance $d = 1/W_{12}$ is the inverse of the weight of the edge connecting the first two vertices in the path.

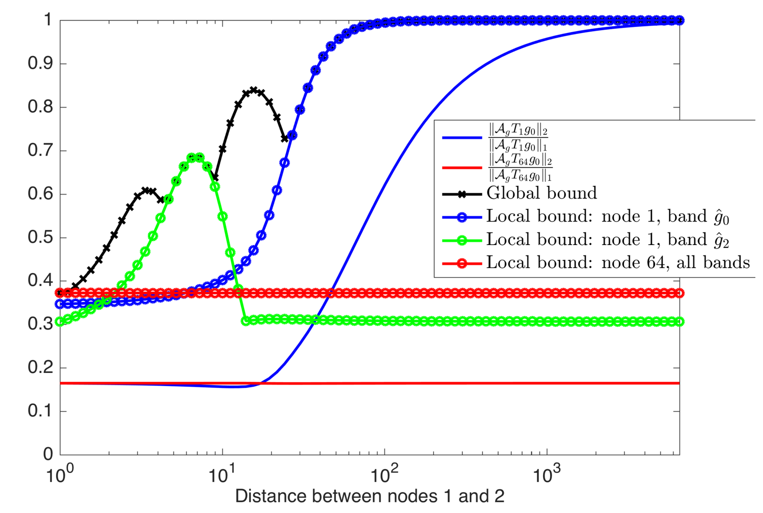


Figure 4: Concentration of the graph Gabor coefficients of $f_1 = T_1 g_0$ and $f_2 = T_{64} g_0$ with respect to the distance between the first two vertices in the modified path graph, along with the upper bounds on this concentration from Theorem 1 (global uncertainty) and Theorem 2 (local uncertainty).

6 Illustrative experiment

Probing the information contained in one sample

- Use the filter $\hat{g}$ corresponding to the frequency band of the data
- Probe the uncertainty using a cheap approximation of our theorem: $i_0 = \tilde{i}$
- Sample inversely proportionally to the uncertainty, i.e: $\propto \|T_i g\|_2$

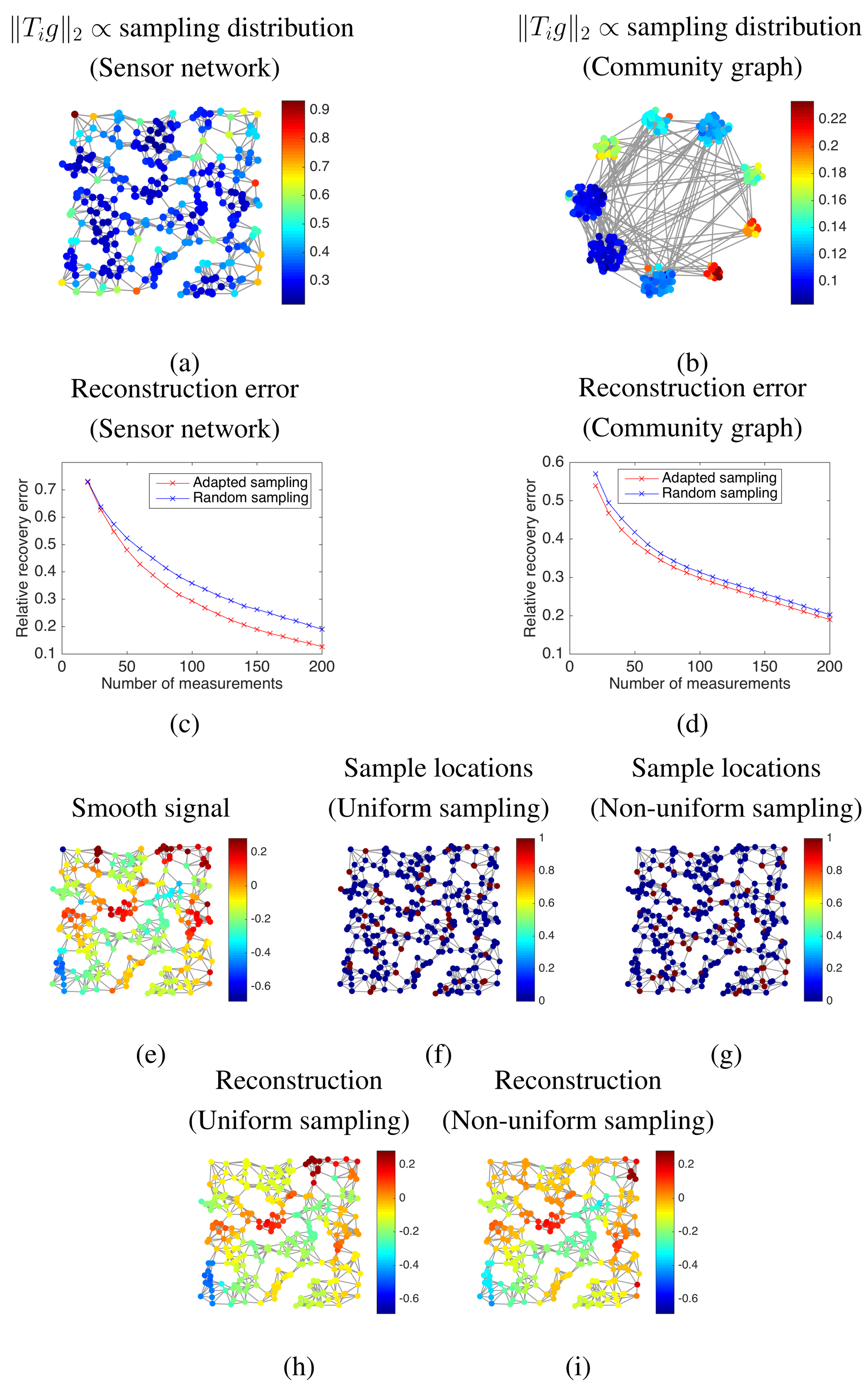


Figure 5: Comparison of random uniform sampling and random non-uniform sampling according to a distribution based on the local sparsity values.

7 Conclusion

Main message

The localization operator $T_i g$ characterizes the local graph structure.

References

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